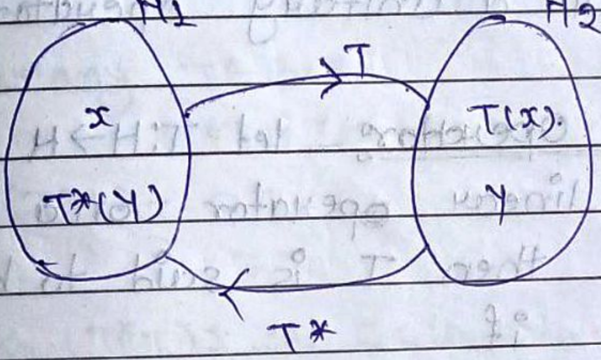
UNIT-V

* **Hilbert adjoint Operator :-** let $T: H_1 \to H_2$ be a bounded linear operator where H_1 and H_2 are Hilbert space then the Hilbert adjoint operator of T is a operator $T^*: H_2 \to H_1$ s.t. $\forall\, x \in H_1,\ y \in H_2$

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$



* **Self Adjoint Operator :-** let $T: H \to H$ be a bounded linear operator of a Hilbert space H then the self adjoint operator is denoted by T^* and define as

$$T^* = T$$

Thus the hilbert adjoint operator T^* of T define as

$$\langle Tx, x\rangle = \langle x, T^*x\rangle$$

because

$$\langle Tx, x\rangle = \langle x, Tx\rangle$$

* **Unitary Operator :-** A bounded linear operator $T: H \to H$ on Hilbert space H is said to be unitary operator if $T^* = T^{-1}$

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and T is bijective. Here

$T^* = T^{-1}$

$\Rightarrow TT^* = TT^{-1}$

and $T^*T = T^{-1}T$

i.e $T^* = T^{-1}$

$\Rightarrow TT^* = I = T^*T$

Thus for a unitary operator T.

10* **Normal Operator:** – let $T: H \to H$ be a bounded linear operator on a Hilbert space H then T is said to be normal operator if

$$T^*T = TT^*$$

10 * **Positive Operator:** – let $T: H \to H$ be a bounded linear operator on a Hilbert space H, then T is said to be positive operator if

$$\boxed{\langle Tx, x\rangle \geq 0} \quad x \in H$$

* **Theorem:** – for any $T \in B(X)$ then T^*T is a positive operator.

Proof: – let $T = 2iI$

Then $Tx = 2iIx$

So $\langle Tx, x\rangle = \langle 2iIx, x\rangle$

$= 2i\langle x, x\rangle$

$= 2i\|x\|^2$

$T = 2iI$

$T^* = -2iI$

$= \langle (T^*T)x, x\rangle = \langle 4Ix, x\rangle$

$= \langle 4x, x\rangle = 4\langle x, x\rangle$

$= 4\|x\|^2$

for any $T \in B(X)$

$T^*T = I$

$\therefore (T^*T)x = Ix$

$= x$

$\therefore \langle (T^*T)x, x\rangle = \langle x, x\rangle$

$= \|x\|^2 \geq 0$

$\langle (T^*T)x, x\rangle \geq 0.$

and $\langle (T^*T)x, x\rangle = \langle T^*(Tx), x\rangle$

$= \langle T(x), T(x)\rangle$

$= \|T(x)\|^2$

$\langle (T^*T)x, x\rangle \geq 0$

$\therefore T^*T$ is a positive operator.

13, 10, 11 * **Theorem:** – $T \in B(X)$ is self adjoint if and only if $\langle Tx, x\rangle$ is a real no. for every $x \in X$ or in particular every positive operator is self adjoint.

Proof: – let T be self adjoint and let $x \in X$ then,

$\langle Tx, x\rangle = \langle x, T^*x\rangle$
$= \langle x, Tx\rangle$
$= \overline{\langle Tx, x\rangle}$

Hence $\langle Tx, x\rangle$ is a real number.

Conversely:- suppose that $\langle Tx, x\rangle$ is a real number then we have to show that T is self adjoint we have,

$4\langle Tx, y\rangle = \langle T(x+y), (x+y)\rangle - \langle T(x-y), (x-y)\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), (x-iy)\rangle$

Taking conjugate and interchanging x and y we get,

$4\langle x, Ty\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), (x-y)\rangle - i\langle T(y+ix), y+ix\rangle + i\langle T(y-ix), y-ix\rangle$

comparing the ~~weight~~ right side of the above two eqn we see that

$\langle Tx, y\rangle = \langle x, Ty\rangle = \langle T^*x, y\rangle$

$\Rightarrow \langle Tx, y\rangle = \langle T^*x, y\rangle$

$\Rightarrow \langle Tx, y\rangle - \langle T^*x, y\rangle = 0$

$\Rightarrow \langle Tx - T^*x, y\rangle = 0$

$\langle (T-T^*)x, y\rangle = 0$

$\Rightarrow T - T^* = 0$

$\boxed{T = T^*}$

Thus $T \in B(X)$ is self adjoint.

* Theorem:- The set of all self-adjoint operator is closed linear subspace of $B(X)$ and therefore is a real Banach space.

Proof:- clearly 0 and I are self adjoint operator. If T_1 and T_2 are self adjoint and α and β are real number then,

$(\alpha T_1 + \beta T_2)^* = (\alpha T_1)^* + (\beta T_2)^*$ {∵ property of adjoint operators}

$= \bar{\alpha} T_1^* + \bar{\beta} T_2^*$

$= \alpha T_1 + \beta T_2$ {∵ self adjoint}

so that $(\alpha T_1 + \beta T_2)$ is self adjoint. Now if $\{T_n\}$ is a sequence of self adjoint operator which converges to T then,

$\|T - T^*\| = \|(T - T_n) + (T_n - T_n^*) + (T_n^* - T^*)\|$

$\leq \|T - T_n\| + \|T_n - T_n^*\| + \|T_n^* - T^*\|$

$= \|T - T_n\| + \|(T_n - T)^*\|$

{∵ T_n is self adjoint}

$= \|T - T_n\| + \|T_n - T\|$

{∵ The Hilbert adjoin operator T^* is unique with the norm $\|T^*\| = \|T\|$}

$= 2\|T_n - T\| \to 0$ as $n \to \infty$

$\boxed{T = T^*}$

Hence, T is self adjoint operator. This complete the proof of theorem.

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Theorem:- (Sequence of self adjoint Operator)

Statement:- let $\{T_n\}$ be a sequence of self adjoint linear operators. $T_n: H \to H$ on hilbert space H. Suppose that $\{T_n\}$ converges say $T_n \to T$ that is $\|T_n - T\| \to 0$ where $\|\cdot\|$ is norm on the space $B(H,H)$ then the limit operator T is self adjoint operator on H.

Proof:- let $\{T_n\}$ be a sequence of bounded self adjoint linear operators $T_n: H \to H$ on Hilbert space H.
suppose that $\{T_n\}$ converges to T.
Then we have to show that
$$T^* = T$$
i.e. $\|T - T^*\| = 0$
we know that "The hilbert adjoint operator T^* of T exists is unique and is a bounded linear operator with norm $\|T^*\| = \|T\|$."
Thus it follows that,
$$\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\| \quad \text{---} (1)$$
Then
$$\|T - T^*\| = \|(T - T_n) + (T_n - T_n^*) + (T_n^* - T^*)\|$$
$$\leq \|T - T_n\| + \|T_n - T_n^*\| + \|T_n^* - T^*\|$$
$$= \|T - T_n\| + \|(T_n - T)^*\|$$
$$= \|T - T_n\| + \|T_n - T\| \quad \{\because \text{by } (1)\}$$
$$= 2\|T - T_n\| \to 0 \text{ as } n \to \infty.$$
Therefore
$$T = T^*$$
This complete the proof of theorem.

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* Theorem:- The set of all normal operator is a closed subset of $B(X)$ which contains the set of all self adjoint operator and closed under scalar multiplication.

Proof:- obviously, every self adjoint operator is normal and if T is normal and α is any scalar then αT is normal.
If $\{T_n\}$ is a sequence of normal operator such that,
$$T_n \to T \text{ as } n \to \infty$$
Then
$$T_n^* \to T^*$$
and so,
$$\|TT^* - T^*T\| = \|(TT^* - T_nT_n^*) + (T_nT_n^* - T_n^*T_n) + (T_n^*T_n - T^*T)\|$$
$$\leq \|TT^* - T_nT_n^*\| + \|T_nT_n^* - T_n^*T_n\| + \|T_n^*T_n - T^*T\|$$
$$\{\because \|T_nT_n^* - T_n^*T_n\| \to 0\}$$

$= \|TT^* - T_nT_n^*\| + \|T_n^*T_n - T^*T\|$

$= 2\|TT^* - T_nT_n^*\|$

Hence.

$$\boxed{TT^* = T_nT_n^*}$$

Thus T is normal.

12 * Theorem :- If T_1 and T_2 are normal operator such that each commutes with the adjoint of other then prove that T_1+T_2 and $T_1.T_2$ are normal.

Proof :- We have T_1 and T_2 are normal operator such that each commutes with the adjoint of the other i.e

$$T_1^*T_2 = T_2T_1^* \Leftrightarrow T_1T_2^* = T_2^*T_1 \quad \text{---} (1)$$

Now,

$$(T_1+T_2)(T_1+T_2)^* = (T_1+T_2)(T_1^*+T_2^*)$$
$$= T_1T_1^* + T_1T_2^* + T_2T_1^* + T_2T_2^* \quad \text{---} (2)$$

and,

$$(T_1+T_2)^*(T_1+T_2) = (T_1+T_2)^*(T_1+T_2)$$
$$= (T_1^*+T_2^*)(T_1+T_2)$$
$$= T_1^*T_1 + T_1^*T_2 + T_2^*T_1 + T_2^*T_2 \quad \text{---} (3)$$

from eqn (1), (2) and (3) we get,

$$(T_1+T_2)(T_1+T_2)^* = (T_1+T_2)^*(T_1+T_2)$$

To prove $T_1.T_2$ is normal.

$$(T_1.T_2)(T_1.T_2)^* = (T_1.T_2)(T_1^*.T_2^*)$$
$$= T_1T_2T_1^*T_2^*$$
$$= T_1^*T_2^*T_1T_2$$
$$= (T_1^*.T_2^*)(T_1.T_2)$$
$$(T_1.T_2)(T_1.T_2)^* = (T_1.T_2)^*(T_1.T_2)$$

Hence T_1+T_2 and $T_1.T_2$ is normal.

12 * Theorem :- $T \in B(X)$ is a normal if and only if its real and imaginary part commutes.

Proof :- let A_1 and A_2 are the real and imaginary parts of T and let $T \in B(X)$ s.t.

$$T = A_1 + iA_2 \quad \text{---} (1)$$

and also, let real and imaginary parts are commutes. i.e.

$$A_1A_2 = A_2A_1 \quad \text{---} (2)$$

Now, we have to show that T is normal for this we will prove

$$TT^* = T^*T \quad \text{---} (3)$$

we have $T = A_1 + iA_2$

and $T^* = A_1 - iA_2$

$$TT^* = (A_1+iA_2)(A_1-iA_2)$$
$$= A_1^2 - iA_1A_2 + iA_1A_2 + A_2^2$$
$$= A_1^2 + A_2^2 \quad \text{---} (4)$$

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$$T^*T = (A_1 - iA_2)(A_1 + iA_2)$$
$$T^*T = A_1^2 + iA_1A_2 - iA_1A_2 + A_2^2$$
$$= A_1^2 + A_2^2 \quad \text{--- (5)}$$

Hence from eqn (4) and (5) we have,

$$TT^* = T^*T$$

$\Rightarrow T$ is normal.

<u>Conversely</u>:- Now suppose that T is normal i.e.

$$TT^* = T^*T$$

then we have to show that real and imaginary parts are commutes we will show

$$A_1A_2 = A_2A_1$$

Now,

$$TT^* = T^*T$$
$$A_1^2 + A_2^2 + i(A_2A_1 - A_1A_2) = A_1^2 + A_2^2 + i(A_1A_2 - A_2A_1)$$
$$i(A_2A_1 - A_1A_2) = i(A_1A_2 - A_2A_1)$$
$$A_2A_1 - A_1A_2 = A_1A_2 - A_2A_1$$
$$A_2A_1 + A_2A_1 = A_1A_2 + A_1A_2$$
$$2A_2A_1 = 2A_1A_2$$
$$A_2A_1 = A_1A_2$$

Hence, real and imaginary parts ~~of~~ commute.

* <u>Theorem</u>:- If $T \in B(X)$ is such that $\langle Tx, x\rangle = 0 \quad \forall x \in X$, then $T = 0$.

<u>Proof</u>:- It is sufficient to show that.

$$\langle Tx, y\rangle = 0 \quad \forall x, y \in X \quad \text{--- (1)}$$

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we have

$$\langle T(\alpha x + \beta y), \alpha x + \beta y\rangle = \langle T\alpha x + T\beta y, \alpha x + \beta y\rangle$$
$$= \langle T\alpha x, \alpha x + \beta y\rangle + \langle T\beta y, \alpha x + \beta y\rangle$$
$$= \langle T\alpha x, \alpha x\rangle + \langle T\alpha x, \beta y\rangle + \langle T\beta y, \alpha x\rangle + \langle T\beta y, \beta y\rangle$$

$$= \alpha\bar{\alpha}\langle Tx, x\rangle + \alpha\bar{\beta}\langle Tx, y\rangle + \beta\bar{\alpha}\langle Ty, x\rangle + \beta\bar{\beta}\langle Ty, y\rangle$$

$$= |\alpha|^2\langle Tx, x\rangle + \alpha\bar{\beta}\langle Tx, y\rangle + \bar{\alpha}\beta\langle Ty, x\rangle + |\beta|^2\langle Ty, y\rangle$$

$$= |\alpha|^2\langle Tx, x\rangle + |\beta|^2\langle Ty, y\rangle + \alpha\bar{\beta}\langle Tx, y\rangle + \bar{\alpha}\beta\langle Ty, x\rangle$$

$$\Rightarrow \langle T(\alpha x + \beta y), \alpha x + \beta y\rangle - |\alpha|^2\langle Tx, x\rangle - |\beta|^2\langle Ty, y\rangle = \alpha\bar{\beta}\langle Tx, y\rangle + \bar{\alpha}\beta\langle Ty, x\rangle$$

$$\Rightarrow 0 = \alpha\bar{\beta}\langle Tx, y\rangle + \bar{\alpha}\beta\langle Ty, x\rangle$$
$$\Rightarrow \alpha\bar{\beta}\langle Tx, y\rangle + \bar{\alpha}\beta\langle Ty, x\rangle = 0 \quad \text{--- (2)}$$

if $\alpha = 1$ and $\beta = 1$ we get

$$\langle Tx, y\rangle + \langle Ty, x\rangle = 0 \quad \text{--- (3)}$$

Put $\alpha = i$, $\beta = 1$

$$i\langle Tx, y\rangle - i\langle Ty, x\rangle = 0 \quad \text{--- (4)}$$
$$\langle Tx, y\rangle - \langle Ty, x\rangle = 0 \quad \{\because i \neq 0\}$$

By eqn (3) and (4)

$\langle Tx, y\rangle + \langle Ty, x\rangle = 0$

$\langle Tx, y\rangle - \langle Ty, x\rangle = 0$

$2\langle Tx, y\rangle = 0$

$\Rightarrow \langle Tx, y\rangle = 0$

$\Rightarrow T = 0$

This complete the proof of theorem

* Theorem :- let $T: H \to H$ be operator then (i) T is normal $\Leftrightarrow \|T^*x\| = \|Tx\|$

(ii) If T is normal then

$\|T^2\| = \|T\|^2$.

Proof :- (i) we have,

$\|T^*x\| = \|Tx\| \Leftrightarrow \|T^*x\|^2 = \|Tx\|^2$

$\Leftrightarrow \langle T^*x, T^*x\rangle = \langle Tx, Tx\rangle$

$\Leftrightarrow \langle TT^*x, x\rangle = \langle T^*Tx, x\rangle$

{by defn of adjoint operator}

$\Leftrightarrow \langle TT^*x, x\rangle - \langle T^*Tx, x\rangle = 0$

$\Leftrightarrow \langle TT^*x - T^*Tx, x\rangle = 0$

$\Leftrightarrow \langle (TT^* - T^*T)x, x\rangle = 0$

$\Leftrightarrow TT^* - T^*T = 0$

$\{\because \langle Tx, x\rangle = 0 \Leftrightarrow T = 0\}$

$\Leftrightarrow TT^* = T^*T$

Hence T is normal.

(ii) If T is normal then from (i) we have

$\|T^2x\| = \|TTx\|$

$= \|T^*Tx\|$ $\{\because$ by (i) $\|T^*x\| = \|Tx\|\ \forall x\}$

Hence. $\|T^2\| = \|T^*T\|$ $\{\because$ by property, $\|T^*T\| = \|T\|^2\}$

$= \|T\|^2$

$$\boxed{\|T^2\| = \|T\|^2}$$

Hence Proved.

Q. 16 Theorem :- $T \in B(X)$ is unitary if and only if it is isometrically isomorphism of X onto itself.

Proof :- suppose T be unitary then

$\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle$ {: defn of adjoint operator}

$= \langle x, y\rangle$ {: T is unitary so $T^*T = I$ $\forall x, y \in X$}

Therefore

$\|Tx\|^2 = \langle Tx, Tx\rangle$

$= \langle x, x\rangle$

$\Rightarrow \|Tx\|^2 = \|x\|^2$

$\Rightarrow \|Tx\| = \|x\|$

so, T preserves norm. It is clearly onto and hence is an isometrically isomorphism.

Conversely :- let T be an isometrically isomorphism of X onto itself. Then

$\langle x, x\rangle = \|x\|^2 = \|Tx\|^2$ $\{\because \|Tx\| = \|x\|\}$

$= \langle Tx, Tx\rangle$

$= \langle T^*Tx, x\rangle$ {by defn of adjoint operator}

$\Rightarrow \langle T^*Tx, x\rangle = \langle x, x\rangle$

$\Rightarrow \langle T^*Tx, x\rangle - \langle x, x\rangle = 0$

$\Rightarrow \langle T^*Tx, x \rangle - \langle x, x \rangle = 0$

$\Rightarrow \langle T^*Tx - Ix, x \rangle = 0$

$\Rightarrow \langle (T^*T - I)x, x \rangle = 0$

$T^*T - I = 0 \quad \{\because \langle Tx, x \rangle = 0 \Leftrightarrow T = 0\}$

$\Rightarrow T^*T = I$

also T^* exist and

$T^* = (T^*T)T^{-1}$

$T^* = IT^{-1}$

$TT^* = TT^{-1} = I$

$TT^* = I$

Hence

$TT^* = T^*T = I$

Hence T is unitary.

Lemma: let T be a bounded linear operator on H. Then $T = 0$ if and only if $\langle Tx, y \rangle = 0$ $\forall x, y \in H$.

Proof: suppose that T is the zero operator. Then we have,

$Tx = 0 \quad \forall x \in H$

so, $\langle Tx, y \rangle = \langle 0, y \rangle = 0 \quad \forall x, y \in H$

Conversely: let

$\langle Tx, y \rangle = 0 \quad \forall x, y \in H$

Taking $y = Tx$ we get

$\Rightarrow \langle Tx, Tx \rangle = 0$

$\Rightarrow \|Tx\|^2 = 0$

$\Rightarrow \|Tx\| = 0 \Rightarrow Tx = 0$

$\Rightarrow T = 0.$

Theorem: If T is an operator on H, then following assertion are equivalent.

(i) $T^*T = I$

(ii) $\langle Tx, Ty \rangle = \langle x, y \rangle \quad \forall x, y$

(iii) $\|Tx\| = \|x\| \quad \forall x$.

Proof: (i) $\Rightarrow$ (ii)

let $T^*T = I$

then $\langle Tx, Ty \rangle = \langle T^*Tx, y \rangle$

$= \langle Ix, y \rangle$

$= \langle x, y \rangle$

(ii) $\Rightarrow$ (iii) put $x = y$ in (ii) then

$\langle Tx, Ty \rangle = \langle x, y \rangle$

$\Rightarrow \langle Tx, Tx \rangle = \langle x, x \rangle$

$\Rightarrow \|Tx\|^2 = \|x\|^2$

$\Rightarrow \|Tx\| = \|x\|$

(iii) $\Rightarrow$ (i) $\quad \|Tx\| = \|x\|$

$\Rightarrow \|Tx\|^2 = \|x\|^2$

$\Rightarrow \langle Tx, Tx \rangle = \langle x, x \rangle$

$\Rightarrow \langle T^*Tx, x \rangle = \langle x, x \rangle$

$\Rightarrow \langle T^*Tx - x, x \rangle = 0$

$\Rightarrow \langle (T^*T - I)x, x \rangle = 0$

$\Rightarrow T^*T - I = 0$

$\Rightarrow T^*T = I$

Projection: A perpendicular projection on a Hilbert space is a bounded linear operator with

$P^2 = P$ and $P^* = P$.

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* **Theorem:-** P is the perpendicular projection on a closed linear subspace M of a hilbert space H if and only if (I-P) is the perpendicular projection $M^{\perp}$.

Proof:- suppose P is a perpendicular projection on M then we have,

$P^2 = P$ and $P^* = P$

we have,

$(I-P)^2 = (I-P)(I-P)$

$= I.I - IP - PI + PP$

$= I - P - P + P$ $\{\because IP = P\}$

$= I - P$ $\{\because P^2 = P\}$

and

$(I-P)^* = I^* - P^*$

$= I - P$ $\{\because I^* = I$ and $P^* = P\}$

Thus I-P is a perpendicular projection in H. Now we shall show that if P is define on M then I-P is define on $M^{\perp}$

for this let H be the range of I-P.

Then $x \in N$,

$(I-P)x = x \quad \forall x \in N.$

i.e. $x - Px = x \quad \forall x \in N$

$\Rightarrow Px = 0 \quad \forall x \in N$

$\Rightarrow M \subseteq M^{\perp}$ — ①

further let,

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$x \in M^{\perp}$ then

$Px = 0$

$\Rightarrow x - Px = x$

$\Rightarrow (I-P)x = x$

which shows that x belongs to the range of (I-P)

i.e. $x \in N$

Hence $M^{\perp} \subseteq M$ — ②

By ① and ② we get

$\boxed{M^{\perp} = M}$

Hence (I-P) is the perpendicular projection on $M^{\perp}$.

Conversely:- suppose that (I-P) is the perpendicular projection on $M^{\perp}$ then by the above argument

$P = I - (I-P)$ is a projection.

on $(M^{\perp})^{\perp} = M$

Hence, P is the perpendicular projection on M

* **Theorem:-** If P is the perpendicular projection on the closed linear subspace of H then $x \in M \Leftrightarrow Px = x \Leftrightarrow \|Px\| = \|x\|$.

Proof:- $x \in M \Leftrightarrow Px = x$ follows from the definition of the operator and M is the range P.

Now, it is trivial that

$Px = x \Rightarrow \|Px\| = \|x\|$

To prove the converse assume that

$\|Px\| = \|x\|$

Now, $x = Px + (I-P)x$

$\Rightarrow \|x\| = \|Px + (I-P)x\|$

$\Rightarrow \|x\|^2 = \|Px + (I-P)x\|^2$

Now, $px \in M$ and since P is a perpendicular projection on M therefore $(I-P)$ is a perpendicular projection on $M^{\perp}$.

Thus,

$(I-P)^* = I-P$

so, Px and $(I-P)x$ are orthogonal vectors

By the pythogorian theorem.

$\|Px + (I-P)x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$

$\Rightarrow \|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$ — (1)

$\Rightarrow \|x\|^2 = \|x\|^2 + \|(I-P)x\|^2$

$M = \{x \in M : Px = x\}$

$\Rightarrow \|(I-P)x\|^2 = 0$

$\Rightarrow (I-P)x = 0$

$\Rightarrow x = Px$

* Theorem:- let T be a bounded self adjoint operator on a real Hilbert space H. Then,

$\|T\| = \sup\{|\langle x, Tx\rangle| : \|x\| = 1\}$

Proof:- set

$M = \sup\{|\langle x, Tx\rangle| : \|x\| = 1\}$

Then for any x with $\|x\| = 1$

$|\langle x, Tx\rangle| \leq \|x\|\|Tx\| \leq \|x\|^2\|T\|$

$|\langle x, Tx\rangle| \leq \;\; = \|T\|$

Hence $\|T\| \geq M$ — (1)

In order to prove

$\|T\| \leq M$

we need to consider the following identity

$4\|Tu\|^2 = \langle T(\beta u + \beta^{-1} Tu), \beta u + \beta^{-1} Tu\rangle - \langle T(\beta u - \beta^{-1} Tu), \beta u - \beta^{-1} Tu\rangle$

This is true for any u and any real $\beta \neq 0$.

It follow from the def^n of M that,

$4\|Tu\|^2 \leq M\|\beta u + \beta^{-1} Tu\|^2 + M\|\beta u - \beta^{-1} Tu\|^2$

By the parallelogram law,

$4\|Tu\|^2 \leq 2M\|\beta^2\|u\|^2 + \beta^{-2}\|Tu\|^2)$

taking.

$$\beta^2 = \frac{\|Tu\|}{\|u\|}$$

reduce to

$\|Tu\| \leq M\|u\|$

This is true for any nonzero u in the space and Hence $\boxed{\|T\| \leq M}$ this proves the Property.

* Theorem:- Prove that an operator T on a Hilbert space H is self adjoint $\Leftrightarrow$ $\langle Tx, x\rangle$ is dual for all x.

Proof:- let T be self adjoint and $x \in X$ then,

$\langle Tx, x\rangle = \langle x, T^*x\rangle$
$= \langle x, Tx\rangle$
$= \overline{\langle Tx, x\rangle}$

$\Rightarrow \langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$

Hence. $\langle Tx, x\rangle$ is a real number.

Conversely:- suppose that $\langle Tx, y\rangle$ is real...

$4\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle$
$+ i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle$

By taking ~~conjugate~~ conjugate and in terms x and y.

$4\langle x, Ty\rangle = \langle T(x+y), x+y\rangle - \langle T(y-x), y-x\rangle -$
$i\langle T(y+ix), y+ix\rangle + i\langle T(y-ix), y-ix\rangle$

Comparing above eqn we get,
$\langle Tx, y\rangle = \langle x, Ty\rangle = \langle T^*x, y\rangle$

* State and prove generalised Lax-Milgram Lemma:-

रफ कापी में अलग है।
आया है।

Statement:- let H be a Hilbert space and $\phi: H\times H \to K$ be a ~~subsequence~~ linear functional on H. suppose that there exist constant $\alpha, \beta > 0$ such that,

$|\phi(x, y)| \le \alpha \|x\|\|y\|$ ——— (1)

and

$\beta\|x\|^2 \le Re\,\phi(x, x)$ $\forall x, y \in X$.

Then $\exists$ a unique $S \in B(H)$ s.t.

$\langle x, y\rangle = \phi(x, Sy)$ $\forall x, y \in H$

and also that S^{-1} exist in $B(H)$

$\|S\| \le \frac{1}{\beta}$ $\Rightarrow$ $\|S^{-1}\| \le \alpha$.

* ~~Sub~~ Sequilinear functional:- let X and Y be linear spaces. A linear mapping $\phi: X\times Y \to K$ is called the sequilinear functional if it is linear in the first variable and conjugate linear in the second variable. i.e.

$\phi(kx_1 + x_2, y) = k\phi(x_1, y) + \phi(x_2, y)$
$\phi(x_1, ky_1 + y_2) = \bar{k}\phi(x_1, y_1) + \phi(x_1, y_2)$.

15* Theorem:- let P and Q be the perpendicular projection on closed linear subspace M and N of a hilbert space H prove that PQ is a perpendicular projection on H iff PQ = QP so that in this case PQ is a perpendicular projection on $M \wedge (M \cap N)$.

Proof:- suppose that PQ is a perpendicular projection we have $P^2 = P$ and $P^* = P$,
$Q^2 = Q$ and $Q^* = Q$ ——— (1)

$\because$ PQ is a perpendicular projection

$\therefore$ $(PQ)^* = PQ$

$\Rightarrow$ $P^*Q^* = PQ$

$\Rightarrow$ $QP = PQ$ {$\because$ by (1)}

Conversely:- suppose that $PQ = QP$ — (2)

we have

$(PQ)^* = Q^*P^* = QP$ {by (1)} — (3)

also $(PQ)^2 = (PQ)(PQ)$

$= (PQ)(QP)$ {∵ by (2)}

$= PQQP \Rightarrow PQ^2P$

$= PQP$

$= P(QP)$

$= PPQ = P^2Q = PQ$ — (4)

from (3) and (4) we have,

$(PQ)^* = (PQ)^2$

Now, it remain to show that PQ is a perpendicular project $M \cap N$.

let the range of PQ be R

let $x \in M \cap N \Rightarrow x \in M$ and $x \in N$.

∴ $Px = x$, $Qx = x$ — (5)

Now, $(PQ)x = P(Qx) = Px$ {by (5)}

$(PQ)x = x$ {by (5)}

Thus $M \cap N \subset R$ — (6)

again let $x \in R$ Then $(PQ)x = x$.

$P[(PQ)x] = Px$

$[PP(Qx)] = Px$

$[P^2Q]x = Px$

$(PQ)x = Px$

$x = Px$ {∵ by (5)}

$\Rightarrow x \in M$ — (7)

Similarly $x \in N$ — (8)

from (7) and (8) we get

$x \in M \cap N$

Hence, $R \subset M \cap N$ — (9)

from eqn (6) and (9) we have

$R = M \cap N$

Hence, The range of PQ in $M \cap N$.

* **Projection Operator:**- let M be a subspace of the Hilbert space H. the operator $P: H \to M$ define by $Px = y$ where y is the projection of x onto M is called the projection operator M is called the range of P.

* **Theorem:**- Projection operator P is linear bounded, self adjoint, idempotent if $P \neq 0$ then $\|P\| = 1$.

Proof:- let P be a projection operator from Hilbert space H onto M.

(1) **P is linear:**- let $x_1, x_2 \in H$ then

$x_1 = y_1 + z_1$

$x_2 = y_2 + z_2$

then $Px_1 = y_1$

and $Px_2 = y_2$

$x_1 + x_2 = (y_1 + y_2) + (z_1 + z_2)$

$P(x_1 + x_2) = y_1 + y_2$

$= Px_1 + Px_2$

Now, $x = y + z$

$Px = y$

$ax = ay + az$

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then $P(\alpha x) = \alpha y$

$= \alpha Px$

$\Rightarrow$ P is linear.

(ii) P is bounded :- let $x = y+z$, $y \in M, z \in M^\perp$

then

$\|x\|^2 = \langle x, x\rangle$

$= \langle y+z, y+z\rangle$

$= \langle y,y\rangle + \langle y,z\rangle + \langle z,y\rangle + \langle z,z\rangle$

$= \|y\|^2 + \langle y,z\rangle + \langle z,y\rangle + \|z\|^2$

$= \|y\|^2 + \|z\|^2$

$\Rightarrow \|y\|^2 \le \|x\|^2$

$\Rightarrow \|y\| \le \|x\|$

$\Rightarrow \frac{\|y\|}{\|x\|} \le 1$

Now, $\|Px\| = \|y\|$

$\therefore \frac{\|Px\|}{\|x\|} = \frac{\|y\|}{\|x\|} \le 1$

$\|P\| \le 1$

$\Rightarrow$ P is bounded.

(iii) P is self adjoint :-

$\langle Px_1, x_2\rangle = \langle y_1, y_2\rangle$

where $x_1 = y_1 + z_1$

$Px_1 = y_1$

$x_2 = y_2 + z_2$

$Px_2 = y_2$

$\Rightarrow \langle Px_1, x_2\rangle = \langle y_1, x_2\rangle$

$= \langle y_1, y_2 + z_2\rangle$

$= \langle y_1, y_2\rangle + \langle y_1, z_2\rangle$

$= \langle y_1, y_2\rangle$

$\therefore \langle Px_1, x_2\rangle = \langle y_1, y_2\rangle$ —— (1)

again,

$\langle x_1, Px_2\rangle = \langle x_1, y_2\rangle$

$= \langle y_1 + z_1, y_2\rangle$

$= \langle y_1, y_2\rangle + \langle z_1, y_2\rangle$

$= \langle y_1, y_2\rangle \quad (\because z_1 \in M^\perp)$

$\langle x_1, Px_2\rangle = \langle y_1, y_2\rangle$ —— (2)

By (1) and (2) we get,

$\langle Px_1, x_2\rangle = \langle x_1, Px_2\rangle$

$\Rightarrow P = P^*$

$\Rightarrow$ P is self adjoint.

(iv) P is idempotent :-

let $x = y+z$

$Px = y$

then $P^2x = P(Px)$

$= Py$

$= y$

$\Rightarrow P^2x = Px \quad \forall x \in H$

$\Rightarrow P^2 = P$

$\Rightarrow$ P is idempotent.

* Orthogonal Projection :- Two projection P and Q on a Hilbert space H are orthogonal to each other if their range R(P) and R(Q) are orthogonal

to each other i.e.

$R(P) \perp R(Q)$

we say that a set $\{P_n\}$ of projection is orthogonal if

(i) $P_i \neq 0 \ \forall i$

(ii) P_i is orthogonal to $P_J \ \forall i \neq J$.

* **Theorem**:- let $\{P_1, P_2, \dots, P_n\}$ be a finite orthogonal set of projections. Then ΣP_i is also a projection.

Proof:- since each P_i is projection operator. therefore P_i is linear, bounded, self adjoint and idempotent.

since each P_i is linear and bounded it is easy to show that ΣP_i is linear and bounded. Also,

$(\Sigma P_i)^* = \Sigma P_i^*$

$= \Sigma P_i$

i.e. ΣP_i is self adjoint.

Now, we shall show that ΣP_i is idempotent for this we first show that if P_i and P_J are orthogonal then,

$P_i P_J = 0 \ \forall i \neq J$

let $x \in H$, $P_J(x) \in R(P_J)$

$R(P_i) \perp R(P_J)$

$P_J(x) \perp R(P_i)$

$P_i(P_J(x)) = 0$

$(P_i P_J)(x) = 0 \quad \forall x \in H$

Thus $P_i P_J = 0 \ \forall i \neq J$

$$\left(\sum_{i=1}^{n} P_i\right)^2 = \left(\sum_{i=1}^{n} P_i\right) \cdot \left(\sum_{J=1}^{n} P_J\right)$$

$$= \sum_{i=1}^{n} P_i^2 + \sum_{\substack{i,J \\ i \neq J}} P_i P_J \quad \left\{\because \begin{array}{l} 1 \le i \le n \\ 1 \le J \le n \end{array}\right\}$$

$$= \sum_{i=1}^{n} P_i^2 + 0$$

$$= \sum_{i=1}^{n} P_i \quad \{\because \text{each } P_i \text{ is idempotent}\}$$

consequently ΣP_i is projection operator.

* **State and Prove Lax-milgram Theorem**

Statement:- let X be a hilbert space and let $\phi(x,y)$ be a bounded complex valued function on the product hilbert space $X \times X$ which satisfies the following properties:

① Sesquelinearly:-

$\phi(x_1\alpha_1 + x_2\alpha_2, y) = \alpha_1 \phi(x_1, y) + \alpha_2 \phi(x_2, y)$

and $\phi(x, \beta_1 y_1 + \beta_2 y_2) = \bar{\beta}_1 \phi(x, y_1) + \bar{\beta}_2 \phi(x, y_2)$

② Boundedness i.e. $\exists$ a positive constant γ s.t.

$|\phi(x,y)| \le \gamma \|x\| \|y\|$

③ Positivity i.e. $\exists$ a positive constant δ s.t.

$Re\, \phi(x,x) \ge \delta \|x\|^2$ Then $\exists$ a uniquely determine

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bounded linear operator S with the bounded linear inverse S^{-1} s.t.

$$\langle x,y\rangle = \phi(x,Sy) \ \forall\ x,y\in X$$

and $\|S\| \le \delta^{-1} = \frac{1}{\delta}$ and $\|S^{-1}\| \le \gamma$.

Proof:- since we have given that X is a hilbert space and $\phi : X \times X \to K$ is sesquelinear functional on X and $|\phi(x,y)| \le \gamma \|x\|\|y\|$ also

$$\mathrm{Re}\,\phi\langle x,x\rangle \ge \delta\|x\|^2$$

Then we have to show that $\exists$ a unique $S\in B(X)$ s.t.

$$\langle x,y\rangle = \phi(x,Sy) \text{ and } \|S\| \le \frac{1}{\delta},\ \|S^{-1}\| \le \gamma$$

Now, since ϕ is bounded

$$|\phi(x,y)| \le \gamma\|x\|\|y\|$$

$$\sup_{\|x\|=\|y\|=1} |\phi(x,y)| \le \gamma \Rightarrow \|\phi\| \le \gamma \quad \text{--- (1)}$$

So, $\exists$ $A\in B(X)$ s.t.

$$\phi\langle x,y\rangle = \langle Ax,y\rangle \ \forall\ x,y\in X \quad \text{--- (2)}$$

moreover

$$\|A\| = \|\phi\| \le \gamma \quad \text{--- (3)}$$

{if $\phi : X_1 \times X_2 \to K$ be a bounded sequentially linear then $\exists$ unique $S\in B(X)$

$$\phi(x,y) = (Sx,y)$$

$$\|\phi\| = \|S\| \}$$

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using the positivity property

$$\delta\|x\|^2 \le \mathrm{Re}\,\phi(x,x)$$

$$\delta\|x\|^2 = |\phi(x,x)|$$

$$\delta\|x\|^2 = |\langle Ax,x\rangle|$$

$$\delta\|x\|^2 \le \|Ax\|\cdot\|x\|$$

Hence $\delta\|x\| \le \|Ax\|$ --- (4)

since $\delta>0$ this shows that $Ax=0 \Leftrightarrow x=0$ and Hence, A is one-one.

The range $R(A)$ is a subspace of X. let $y\in \overline{R(A)}$ then there is a sequence $\langle y_n\rangle$ in $R(A)$ s.t. $y_n\to y$ and $y_n = A(x_n)$, $x_n\in X$ and so

$$\delta\|x_n - x_m\| \le \|A(x_n - x_m)\|$$

$$= \|y_n - y_m\|$$

$\because \delta>0$ this shows that $\langle x_n\rangle$ is a cauchy sequence in X and Hence convergence.

If $x_n\to x$ then $y_n \to A(x_n) \to A(x)$ and therefore, $y = A(x)\in R(A)$

Thus, $R(A)$ is closed subspace of X.

Then $x = x_1 + x_2$ where $x_1\in R(A)$, $x_2\in R(A)^{\perp}$ also we have

$$\delta\|x_2\|^2 \le \mathrm{Re}\,\phi\langle x_2,x_2\rangle$$

$$= \mathrm{Re}\,\phi(Ax_2,x_2) = 0$$

$\because Ax_2\in R(A)$ and $x_2\in R(A)^{\perp}$ this shows that $x_2 = 0$ and Hence, $x = x_1\in R(A)$

Thus we see that

$$\boxed{R(A) = X}$$

i.e. $A : X\to X$ is bijective bounded linear

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mapping from eqn (4) we see that
$\delta\|A^{-1}x\| \le \|AA^{-1}x\| = x \quad \forall x \in X$.
Hence A^{-1} is bounded and $\|A^{-1}\| \le 1/\delta$
Thus, A^* and $(A^{-1})^*$ exist in $R(A)$ and
$A^*(A^{-1})^* = (A^{-1}A)^* = I^*$.